Homework 1: Solutions to exercises not appearing in Pressley (also 1.2.3).

Math 120A

- (1.11) (i) Let $\gamma(t) = (1 + \cos t, \sin t(1 + \cos t)$. We set $x = 1 + \cos t$. This implies that $\sin t = \pm\sqrt{2x - x^2}$. Hence we have $y = \pm x\sqrt{2x - x^2}$, or $y^2 = x^2(2x - x^2)$ as a Cartesian equation for the curve.

  (ii) Let $\gamma(t) = (t^2 + t^3, t^3 + t^4)$. We first observe that $y = tx$. To exploit this relationship, consider $x^4 = x(x^3) = (t^2 + t^3)(x^3) = t^2x^3 + t^3x^3 = y^2x + y^3$. Ergo the points of our parametrization satisfy $x^4 = y^2x + y^3$. To see that the parametrized curve is the entire set of solutions to this equation, first note that if $x = 0$, then $y = 0$. If (x, y) is a solution to the equation such that $x \neq 0$, write $y = tx$, where t is whatever constant makes this true. Then solving for x shows there is a unique solution to the equation that satisfies $y = tx$ for any particular t, and it is (t^2+t^3, t^3+t^4).

- (1.12) (i) The tangent vector is $\dot\gamma(t) = (-\sin t, \cos t(1+\cos t)-\sin^2 t) = (-\sin t, \cos t + \cos^2 t - \sin^2 t)$. This vanishes when $t = n\pi$ for n odd.

  (ii) The tangent vector is $\dot\gamma(t) = (2t + 3t^2, 3t^2 + 4t^3)$, which vanishes when both $t(2 + 3t) = 0 = t^2(3 + 4t)$. This happens only when $t = 0$.

- (1.15) We have $\gamma(t) = (e^t \cos t, e^t \sin t)$, so $\dot\gamma(t) = (e^t(\cos t-\sin t), e^t(\sin t+\cos t))$. Their dot product is then $\gamma(t) \cdot \dot\gamma(t) = e^{2t}(\cos^2 t - \cos t \sin t + \sin^2 t + \cos t \sin t) = e^2t$. Now, $||\gamma(t)|| = e^t$, and $||\dot\gamma(t)|| = \sqrt{e^2t(\cos^2 t - 2\sin t\cos t + \sin^2 t + \sin^2 t + 2\sin t\cos t + \cos^t)} = \sqrt{2e^{2t}}$. Hence if the angle between $\gamma(t)$ and $\dot\gamma(t)$ is θ, we have $\cos\theta = \frac{\gamma(t)\cdot\dot\gamma(t)}{||\gamma(t)||||\dot\gamma(t)||} = \frac{1}{\sqrt{2}}$. Ergo $\theta = \frac{\pi}{4}$ (where the sign is determined from a look at the graph).

- (1.2.3) Our curve is $\gamma(\theta) = (r(\theta)\cos\theta, r(\theta)\sin\theta)$, for r a smooth function. The tangent vector $\dot\gamma(\theta) = (r'(\theta)\cos\theta - r(\theta)\sin\theta, r'(\theta)\sin\theta + r(\theta)\cos\theta)$. For γ to be regular, we need both components of the tangent vector to be nonzero everywhere. A computation shows that $||\dot\gamma(\theta)||^2 = r(\theta)^2 + r'(\theta)^2$, so the tangent vector can only be zero if $r = r' = 0$ at some θ; as long as this never happens, the curve is regular. For γ to be unit speed, we need $r(\theta)^2 + r'(\theta)^2 = 1$. An obvious solution is $r \equiv \pm 1$, which gives the unit circle with some orientation. For the other solution, we separate variables to solve the differential equation:

$$
\begin{aligned}
1 &= (\frac{dr}{d\theta}^2) + r^2 \\
\pm\sqrt{1-r^2} &= \frac{dr}{d\theta} \\
\pm d\theta &= \frac{dr}{\sqrt{1-r^2}} \\
\pm\int d\theta &= \int \frac{dr}{\sqrt{1-r^2}} \\
\pm\theta + C &= \sin^{-1}(r) \\
\sin(\pm\theta + C) &= r
\end{aligned}
$$

Ergo we have $\gamma(t) = (\sin(\pm\theta + C)\cos\theta, \cos(\pm\theta + C)\sin\theta)$. This is a circle of radius $\frac{1}{2}$. For $\theta + C$, the center is $(\frac{1}{2}\sin C, \frac{1}{2}\cos C)$; for $theta + C$, the center is $(\frac{1}{2}\sin C, -\frac{1}{2}\cos C)$ (This can be checked by showing that the parametrization satisfies the equation of the appropriate circle. A graphical argument was also ok.)

- Question 4. We see that

$$
\begin{aligned}
\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} \\
&= \lim_{h\to 0} \frac{0-0}{h} \\
&= 0
\end{aligned}
$$

Similarly, $\frac{\partial f}{\partial y}(0,0) = 0$. However, if $\{(x_n, y_n)\}$ is any sequence of points such that $x_n = y_n^2$ and $y_n \to 0$, then $f(x_n, y_n) = \frac{y_n^4}{2y_n^4} = \frac{1}{2}$, so as $n \to \infty$, $f(x_n, y_n) \to \frac{1}{2} \neq f(0,0)$. Ergo f is not continuous.